WHAT'S IN *YOUR* PURSE?

by Geoffrey H. Morley (MathsJam 2015)

Which spending strategy and set of a given number of coin denominations would minimise the average number of coins you carry?

Assumptions

* Prices modulo the lowest denomination banknote are distributed uniformly.

* The cashier has an unlimited supply of coins and change is given using the fewest possible coins.

A SPENDING STRATEGY is a set of rules that determines which coins to hand over, given the price and the coins you have.

THE MINIMALIST SPENDER

Spends coins so as to minimise the number of coins in their purse after each transaction.

Price: 64p.   Purse: 5p, £1.

Would you hand over just the £1 coin, or both coins to end up with two fewer coins?

The Minimalist Spender is the optimal strategy but not the most realistic.

The GREEDY ALGORITHM for giving change is to repeatedly give the the largest denomination not exceeding the amount owed.

Let A be the sum of the two largest denominations. If the greedy decomposition of every amount less than A is a minimal decomposition, then the greedy decomposition of every amount is a minimal decomposition.

(Theorem by Kozen & Zaks)

Such a coin set is called a greedy coin set. Most currencies use a greedy coin set.

COIN CHANGE PROBLEM

What are the fewest coins that achieve a desired change amount?

Denominations: 1, 3, 4.
6 = ?

Greedy solution:
3 coins: 4 + 1 + 1

Optimal solution:
2 coins: 3 + 3   MinCoins(6) = 2

MinCoins(6) is the smallest of the following:

1 + MinCoins(5)   #give 1

1 + MinCoins(3)   #give 3

1 + MinCoins(2)   #give 4

We can calculate the fewest coins needed if we know the fewest coins for every smaller amount.

COIN CHANGE SOLUTION

```
Dim D() As Integer= {1,5,8,20,31,33}
Dim MinCoins(99) As Integer
Dim i, m, p As Integer
For p = 1 to 99
    MinCoins(p) = 9999     ’no solution
    For i = 0 to 5
        If D(i) > p Then Exit For
        m = MinCoins(p - D(i)) + 1
        If m < MinCoins(p) _
        Then MinCoins(p) = m
    Next i
    Console.WriteLine(CStr(p) + " " _
    + CStr(MinCoins(p)))
Next p
```

Denominations are in ascending sequence.

Every change amount is possible if D(0)=1.

ALTERNATIVE SPENDING STRATEGIES

THE COIN KEEPER*

Pays no coins at all.

THE BIG SPENDER*

Pays by banknotes if insufficient change, else overpays as little as possible, breaking ties by favouring bigger denominations.

THE GREEDY BIG SPENDER**

Pays by banknotes if insufficient change, else pays greedily overpaying as little as possible.

*: Pudwell and Rowland

**: Morley

GREEDY BIG SPENDER: RESULT OF SIMULATION

(100 million random prices)

	Estimated proportion of coins	
	In circulation 31 March 2014	Held by greedy big spenders
£2	1.4%	2.9%
£1	5.4%	3.7%
50p	3.3%	5.2%
20p	9.5%	11.5%
10p	5.6%	8.0%
5p	13.3%	12.1%
2p	22.6%	28.2%
1p	38.9%	28.4%

	Average coins	Max. coins
Minimalist Spender	4.6	8
Greedy Big Spender	15.0	65

The Minimalist Spender

Optimal Coin Sets

Optimal Greedy Coin Sets

PRICES 0 TO 99 (× 5p)

4 denominations	AVERAGE COINS (MAXIMUM COINS)
1, 5, 18, 25/29	3.89(6)
1, 3, 11, 37/38	4.1(7)

5 denominations	
1, 5, 16, 23, 33	3.29(6)
1, 3, 7, 18, 44	3.46(6)
1, 3, 8, 20, 44	3.46(6)

6 denominations

Current coinage without 1p, 2p:

1, 2, 4, 10, 20, 40	3.4(6)
1, 4, 6, 21, 30, 37	2.92(4)
1, 5, 8, 20, 31, 33	2.92(5)
1, 2, 5, 11, 26, 66	3.14(5)

PRICES 0 TO 99 (× 5p) (cont.)

7 denominations

1, 4, 9, 11, 26, 38, 44 — 2.65(4)

1, 2, 5, 8, 18, 28, 65/66
1, 2, 5, 8, 19, 30, 63/66 } 2.86(4)

PRICES 0 TO 199 (× 5p)

7 denominations

1, 4, 9, 29, 44, 63, 65 — 3.265(5)

1, 2, 5, 13, 21, 48, 123
1, 2, 5, 13, 21, 49, 124/125/126
1, 2, 5, 13, 21, 50, 124/127 } 3.56(5)

8 denominations

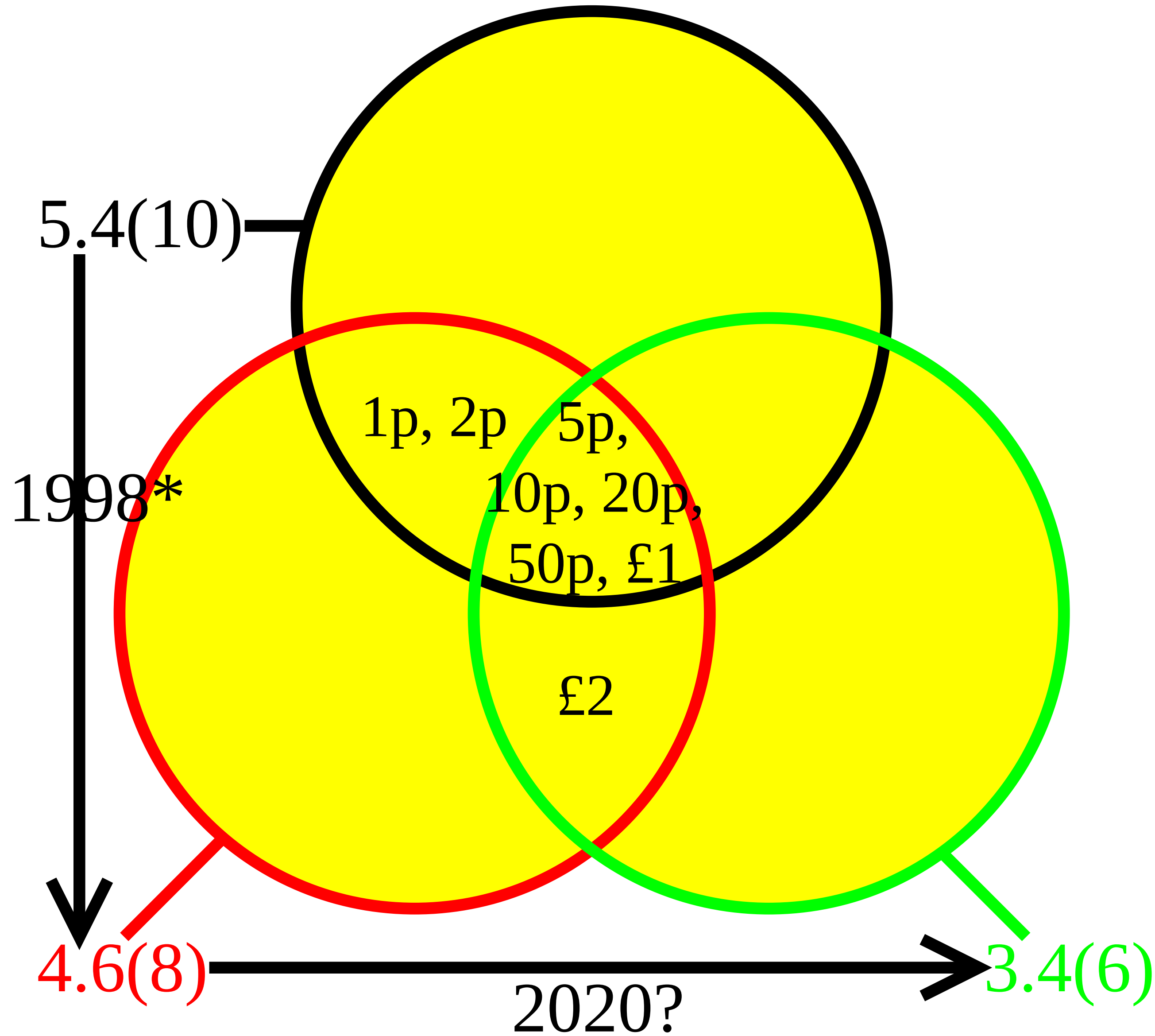
1, 5, 8, 22, 24, 55, 81, 87 — 3.03(4)

1, 2, 5, 8, 17, 38, 59, 131 — 3.31(5)

Almost-optimal greedy coin set:

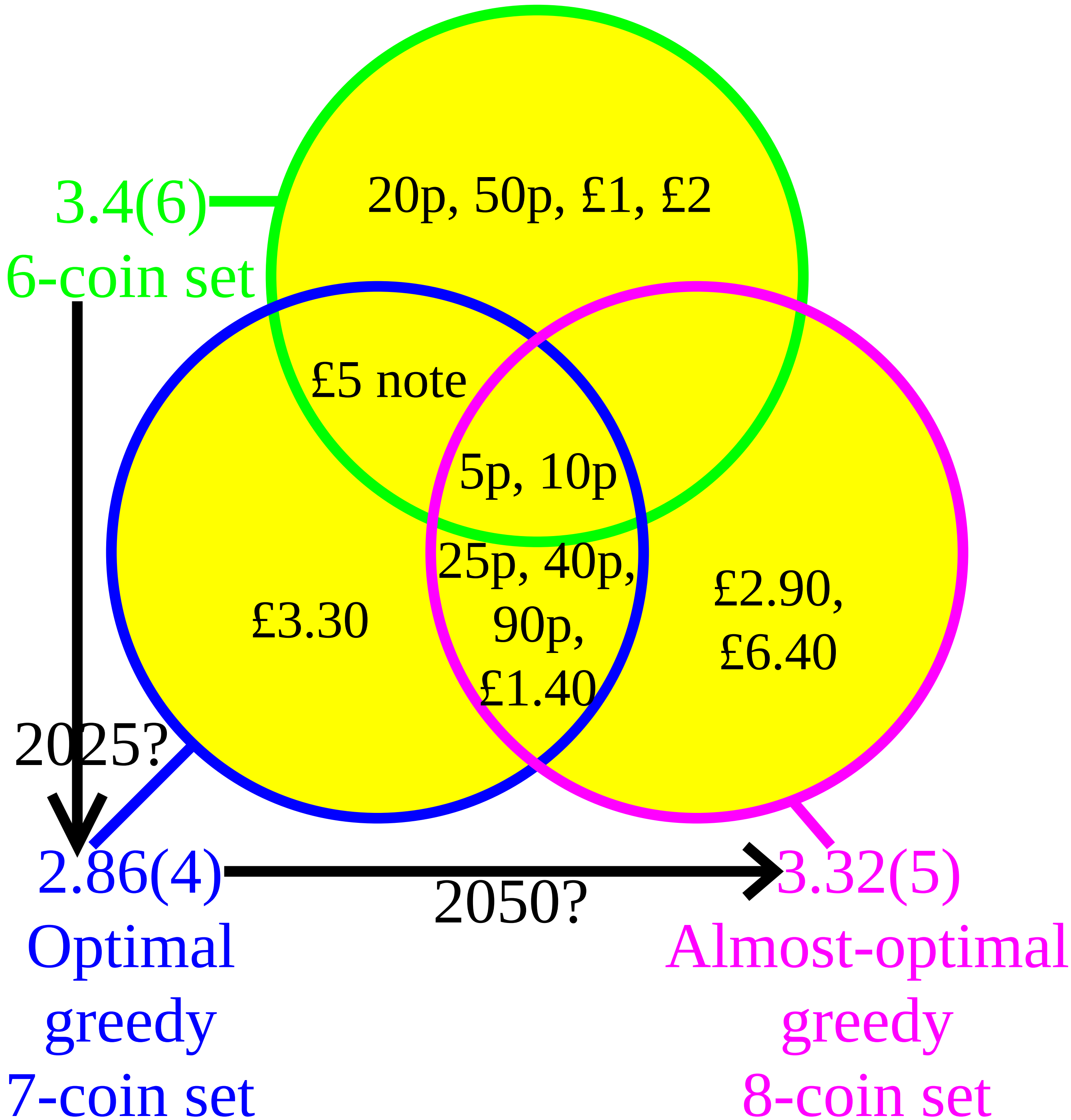
1, 2, 5, 8, 18, 28, 58, 128 — 3.32(5)

WHERE ARE WE?



*: £2 coins dated 1997 first circulated in 1998.

WHERE COULD WE GO?



FURTHER READING

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